Total Marks – 100
Attempt Questions 1-4
All questions are of equal value

Question 1 *(25 marks)* Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\int_0^3 \frac{2dx}{\sqrt{x^2+16}}$. **2**

(b) Find $\int \frac{\sin x}{\cos^3 x}dx$. **2**

(c) Find $\int \sin^3 x dx$. **3**

(d) (i) Prove $2\cos A \sin B = sin(A+B) - \sin(A-B)$. **1**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{12}} \cos 4x \sin 2x \, dx$. **2**

(e) Find, by completing the square $\int \frac{1}{x^2+6x+10}dx$. **2**

(f) (i) Find real numbers A, B, C such that

$$\frac{3x^2-x+8}{(1-x)(x^2+1)} = \frac{A}{1-x} + \frac{Bx+C}{x^2+1}$$ **3**

(ii) Hence find $\int \frac{3x^2-x+8}{(1-x)(x^2+1)}dx$. **2**

(g) Evaluate, $\int_0^{\frac{1}{2}} \tan^{-1} x dx$ using integration by parts. **4**

(h) Use the substitution $t = \tan\frac{x}{2}$ to find $\int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x - 4\sin x}dx$. **4**

Question 2 *(25 marks)* Use a SEPARATE writing booklet. **Marks**

(a) If $z = 3 - 4i$ and $w = 2 + 5i$,
Express in the form $(x + iy)$ where x and y are real.

(i) zw **1**

(ii) $\overline{zw}$ **1**

(iii) $\sqrt{z}$ **2**

(iv) $\dfrac{z}{w}$ **2**

(b) (i) Express in modulus argument form, $\sqrt{3} + i$. **1**

(ii) Hence evaluate $\left(\sqrt{3} + i\right)^5$ in the form $x + iy$. **3**

(c) Graph the region in the Argand diagram which satisfies $\left|z + \overline{z}\right| \le 2$ and $\left|z - i\right| \le 2$ simultaneously. **3**

(d) Let ω be on of the non real cube roots of 1.

(i) Show $1 + \omega + \omega^2 = 0$. **1**

(ii) Hence find the value of $(2 - \omega)\left(2 - \omega^2\right)\left(2 - \omega^4\right)\left(2 - \omega^5\right)$. **2**

(e) By applying de Moivre's Theorem and by also expanding $(\cos\theta + i\sin\theta)^5$ express $\cos 5\theta$ as a polynomial in $\cos\theta$. **4**

(f) The origin O and points A, B, C representing the numbers z, $\dfrac{1}{z}$ and $z + \dfrac{1}{z}$ respectively are joined to form a quadrilateral. Write down the condition or conditions for z so that the quadrilateral OABC will be

(i) a rhombus **1**
(ii) a square **1**

(g) (i) Find all solutions of $z^6 = -1$. **2**

(ii) Plot the solutions on the Argand diagram, indicating z_1. **1**

Question 3 *(25 marks)* Use a SEPARATE writing booklet. **Marks**

(a) Draw neat sketches of the each following, clearly indicating any intercepts, asymptotes, endpoints, turning points and discontinuities. Make each separate one-third page.

(i) $f(x) = 4 - x^2$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y^2 = f(x)$ **2**

(v) $y = |f(x)|$ **2**

(vi) $y = \left[f(x)\right]^2$ **2**

(vii) $|y| = f(x)$ **2**

(viii) $y = f(|x|)$ **2**

(ix) $y = \log_e f(x)$ **2**

(x) $y = f(e^x)$ **2**

(b) Find the gradient of the tangent to the curve, $x^2 - xy + y^3 = 5$ at the point (1, –2). **2**

(c) (i) State the domain and range of $y = \sin^{-1}(e^x)$. **2**

(ii) Sketch the graph of $y = \sin^{-1}(e^x)$ showing clearly the co-ordinates of any endpoints and the equation of any asymptotes. **2**

Question 4 *(25 marks)* Use a SEPARATE writing booklet. **Marks**

(a) Given that $3-i$ is a zero of the function $P(x)=x^3+ax^2+bx-10$, where and b are real, find the other zeros and the values of a and b. **4**

(b) (i) Find all the roots of the equation $x^4+x^3-3x^2-5x-2=0$, given there is a root of multiplicity 3. **4**

(ii) Sketch $y=x^4+x^3-3x^2-5x-2$, showing all intercepts. **2**

(c) The roots of a certain cubic equation are α, β and γ. Given the following **2**

$$\alpha+\beta+\gamma=-3$$
$$\alpha^2+\beta^2+\gamma^2=29$$
$$\alpha\beta\gamma=-6$$

Form the cubic equation whose roots are α, β and γ.

(d) Factorise $P(x)=x^4-5x^3+4x^2+2x-8$ over

(i) R (all real numbers) **3**

(ii) C (all complex numbers) **2**

(e) The equation $x^3+2x^2+bx-16=0$ has roots α, β, γ such that. $\alpha\beta=4$

(i) Show that $b=-20$. **2**

(ii) Find the cubic equation with roots given by α^2, β^2 and γ^2. **3**

(iii) Find the value of $\alpha^3+\beta^3+\gamma^3$. **3**

End of paper

Year 12 Extension 2 Mathematics Mini 2008

Question 1

a) $\int_0^3 \frac{2}{\sqrt{x^2+16}} = 2\ln\left[x+\sqrt{x^2+16}\right]_0^3$

$= 2\ln\left[3+\sqrt{3^2+16}\right] - 2\ln\left[0+\sqrt{0+16}\right]$

$= 2\ln 8 - 2\ln 4$

$= 2\ln\left(\frac{8}{4}\right)$

$= 2\ln 2$

b) $\int \frac{\sin x}{\cos^3 x}\,dx = -\int \cos^{-3}x\,(-\sin x)\,dx$

$= \frac{1}{2}\cos^{-2}x + c$

$= \frac{1}{2\cos^2 x} + c$

$= \frac{1}{2}\sec^2 x + c$

c) $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx$

$= \int (1-\cos^2 x)\sin x\,dx$

$= \int (\sin x - \cos^2 x \sin x)\,dx$

$= -\cos x + \frac{1}{3}\cos^3 x + c$

$= \frac{1}{3}\cos^3 x - \cos x + c$

d) (i) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

L.HS $= \sin A\cos B + \cos A \sin B - (\sin A \cos B - \cos A \sin B)$

$= 2\cos A \sin B$

$= $ RHS

$\therefore 2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(ii) $\int_0^{\frac{\pi}{12}} \cos 4x \sin 2x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{12}} (\sin 6x - \sin 2x)\, dx$

$$= \frac{1}{2}\left[\frac{-1}{6}\cos 6x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{12}}$$

$$= \frac{1}{2}\left\{\left[-\frac{1}{6}\cos\frac{\pi}{2} + \frac{1}{2}\cos\frac{\pi}{6}\right] - \left[-\frac{1}{6}\cos 0 + \frac{1}{2}\cos 0\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{\sqrt{3}}{4} - \left[-\frac{1}{6} + \frac{1}{2}\right]\right\}$$

$$= \frac{\sqrt{3}}{8} - \frac{1}{6}$$

e) $\int \frac{1}{x^2+6x+10}\, dx = \int \frac{dx}{(x+3)^2+1}$

$$= \tan^{-1}(x+3) + c$$

f) $\frac{3x^2 - x + 8}{(1-x)(x^2+1)} = \frac{A}{1-x} + \frac{Bx+C}{x^2+1}$

$\therefore 3x^2 - x + 8 = A(x^2+1) + (Bx+C)(1-x)$

Put $x = 1$ $\quad 10 = 2A$

$A = 5$

Equating coefficients of x^2 $\quad 3 = A - B$

$3 = 5 - B$

$B = 2$

Equating constants $\quad 8 = A + C$

$8 = 5 + C$

$C = 3$

$\therefore A = 5,\ B = 2,\ C = 3$

$$\int \frac{3x^2 - x + 8}{(1-x)(x^2+1)}\, dx = \int \left(\frac{5}{(1-x)} + \frac{2x+3}{x^2+1}\right) dx$$

$$= \int \left(\frac{5}{1-x} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\right) dx$$

$$= -5\ln|1-x| + \ln(x^2+1) + 3\tan^{-1}x + c$$

$$= \ln\left[\frac{(x^2+1)}{|1-x|^5}\right] + 3\tan^{-1}x + c$$

g) $\int_0^{\frac{1}{2}} \tan^{-1}x \, dx$

Let $u = \tan^{-1}x \quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{1+x^2} \quad v = x$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int_0^{\frac{1}{2}} \tan^{-1}x \, dx = \left[x \tan^{-1}x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{1+x^2} dx$$

$$= \left[x \tan^{-1}x\right]_0^{\frac{1}{2}} - \frac{1}{2}\left[\ln(x^2+1)\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2} - \frac{1}{2}\left(\ln\frac{5}{4} - \ln 1\right)$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2} - \frac{1}{2}\ln\frac{5}{4}$$

h) $\int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x - 4\sin x} dx$

$t = \tan\frac{x}{2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(1+t^2)$

$dx = \frac{2}{1+t^2} dt$

when $x = 0$, $t = \tan\frac{x}{2}$

$= 0$

when $x = \frac{\pi}{2}$, $t = \tan\frac{\frac{\pi}{2}}{2}$

$= 1$

$$= \int_0^1 \frac{1}{5+3\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right)} \times \frac{2}{1+t^2} dt$$

$$= \int_0^1 \frac{2\,dt}{5(1+t^2)+3(1-t^2)-4(2t)}$$

$$= \int_0^1 \frac{2\,dt}{5+5t^2+3-3t^2-8t}$$

$$= \int_0^1 \frac{2\,dt}{2(t^2-4t+4)}$$

$$= \int_0^1 \frac{dt}{(t-2)^2}$$

$$= \left[\frac{-1}{t-2}\right]_0^1$$

$$= \left[\frac{-1}{1-2}\right] - \left[\frac{-1}{0-2}\right]$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

Question 2

a) $z = 3-4i\ ,\quad w = 2+5i$

(i) $zw = (3-4i)(2+5i)$
$= 26+7i$

(ii) $\overline{zw} = 26-7i$

(iii) $x+iy = \sqrt{z}$
$(x+iy)^2 = 3-4i$
$x^2-y^2 = 3$
$2xy = {}^-4$
$xy = -2$
$x = \pm 2\ ,\ y = \mp 1$
$\therefore \sqrt{z} = 2-i\ ,-2+i$

(iv) $\dfrac{z}{w} = \dfrac{3-4i}{2+5i}$

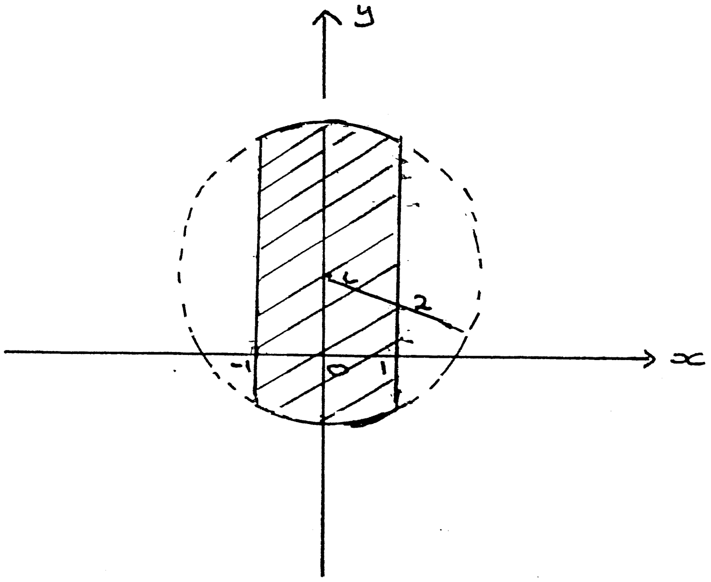
$= \dfrac{3-4i}{2+5i} \times \dfrac{2-5i}{2-5i}$

$= \dfrac{-14-23i}{29}$

b (i) $z = \sqrt{3}+i$
$|z| = \sqrt{(\sqrt{3})^2+1^2}$
$= 2$
$\arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad -\pi < \theta \leq \pi$
$= \frac{\pi}{6}$
$\therefore \sqrt{3}+i = 2 \operatorname{cis} \frac{\pi}{6}$

(ii) $(\sqrt{3}+i)^5 = 2^5 \operatorname{cis} \frac{5\pi}{6}$
$= 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
$= 32\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$
$= -16\sqrt{3} + i16$

c)



$|z+\bar{z}| \le 2$

$|x+iy+x-iy| \le 2$

$|2x| \le 2$

$|x| \le 1$

d) i) $\omega^3 = 1$

$\omega^3 - 1 = 0$

$(\omega-1)(\omega^2+\omega+1) = 0$

Since $\omega - 1 \neq 0$, $\omega \neq 1$ ω is not rational

$\omega^2+\omega+1 = 0$

(ii) $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5)$ $\omega^3 = 1$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$= [(2-\omega)(2-\omega^2)]^2$

$= (4-2\omega-2\omega^2+\omega^3)^2$ $\omega^3=1$

$= [5-2(\omega+\omega^2)]^2$ $\omega^2+\omega+1=0$

$\omega^2+\omega=-1$

$= (5-2(-1))^2$

$= 7^2$

$= 49$

e) $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i^2\sin^2\theta)$
$+ 10\cos^2\theta(i^3\sin^3\theta) + 5\cos\theta(i^4\sin^4\theta) + (i^5\sin^5\theta)$

$\cos 5\theta + i\sin 5\theta = \cos^5\theta + 5\cos^4(i\sin\theta) - 10\sin^2\theta\cos^3\theta$
$- 10\cos^2\theta(i\sin\theta) + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating real parts

$\cos 5\theta = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\cos\theta\sin^4\theta$

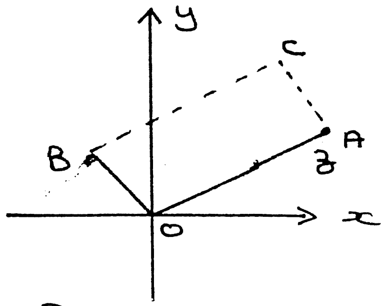
$\therefore \cos 5\theta \quad = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10(1-\cos^2\theta)\cos^3\theta + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$



f) (i) $|z| = 1$

(ii) $|z| = 1$ and $\arg z = \frac{\pi}{2}$

$|z| = 1$, $|z + \frac{1}{z}| = |z - \frac{1}{z}|$

or $\arg(\frac{1}{z}) - \arg z = \frac{\pi}{2}$ or $z^2 + \frac{1}{z^2} = 0$

g) $z^6 = -1$

Let $r(\cos\theta + i\sin\theta) = \sqrt[6]{-1}$

$r^6(\cos 6\theta + i\sin 6\theta) = -1$

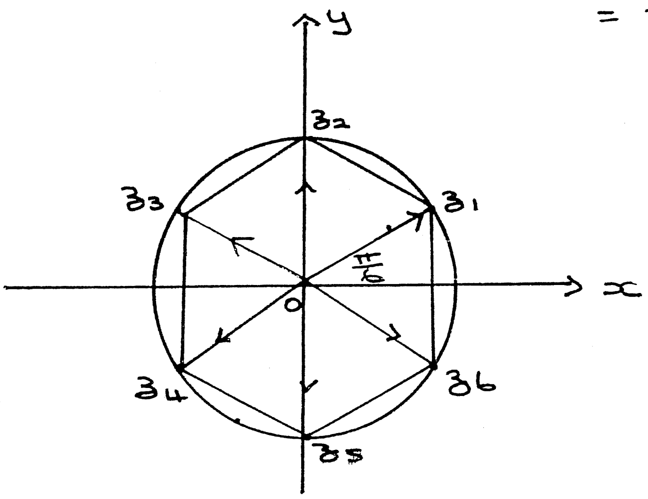
$\cos 6\theta = -1$

$6\theta = (2k+1)\pi$

$\theta = \frac{(2k+1)\pi}{6} \quad k = 0, 1, 2, 3, 4, 5$

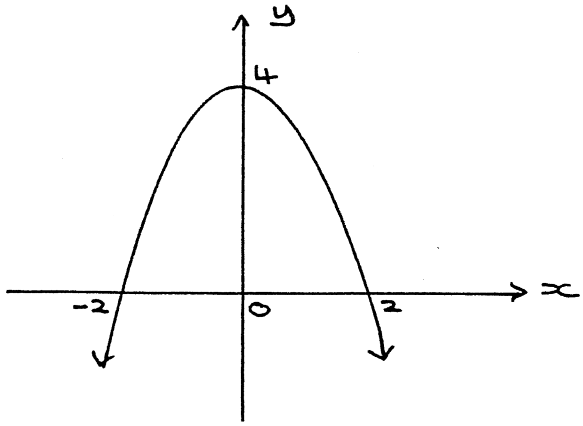
$= \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6} = -\frac{5\pi}{6}$

$\frac{9\pi}{6} = -\frac{\pi}{2}, \quad \frac{11\pi}{6} = -\frac{\pi}{6}$

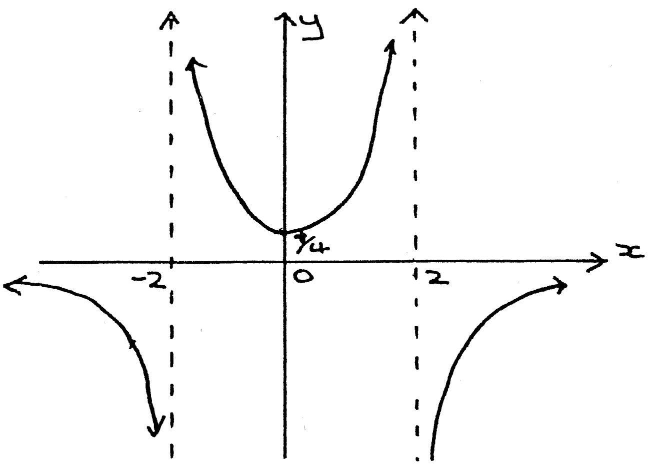


Question 3

a) i $\quad y = 4 - x^2$

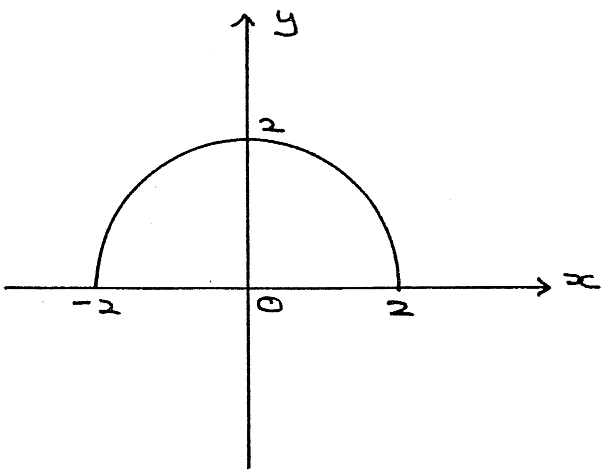


(ii) $y = \frac{1}{f(x)} \qquad x \neq \pm 2$



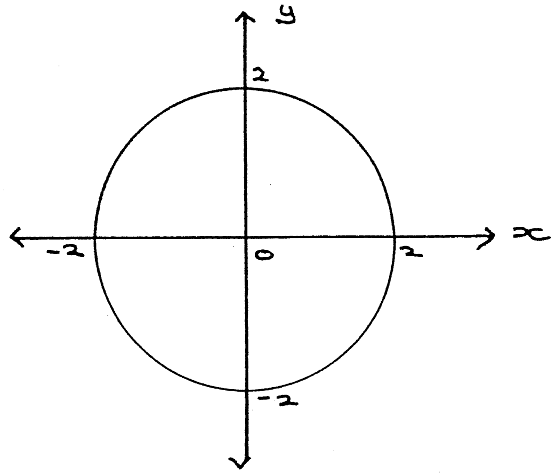
(iii) $y = \sqrt{f(x)} \qquad 4 - x^2 \geq 0$

$-2 \leq x \leq 2$

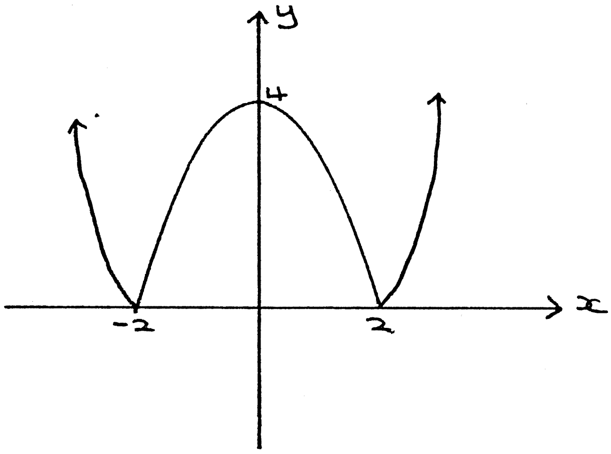


(iv) $y^2 = f(x)$

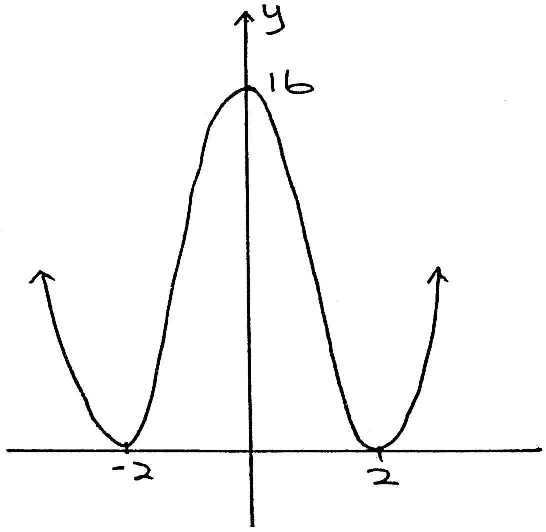
$y = \pm\sqrt{f(x)}$



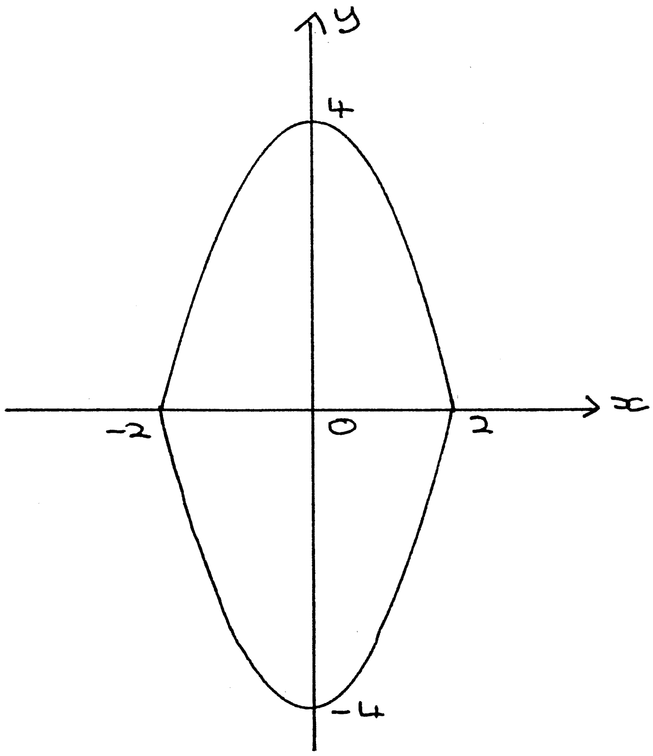
v) $y = |f(x)|$



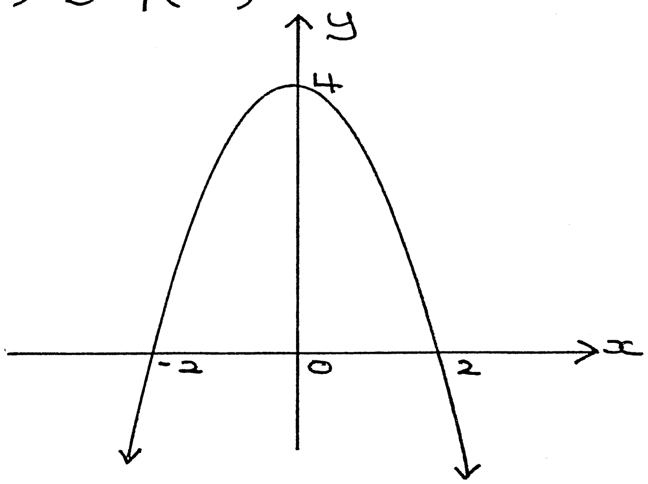
vi) $y = [f(x)]^2$



vii) $|y| = f(x)$



viii) $y = f(|x|)$



ix) $y = \log_e f(x)$

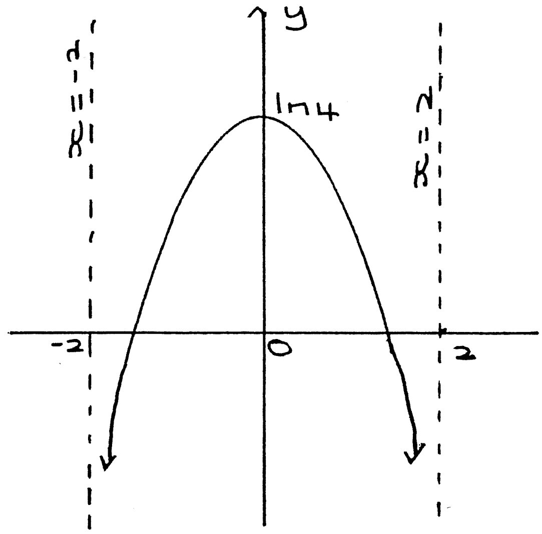
$f(x) > 0$

x-intercept $\Rightarrow y = 0$

$\therefore\ 4 - x^2 = 1$

$x^2 = 3$

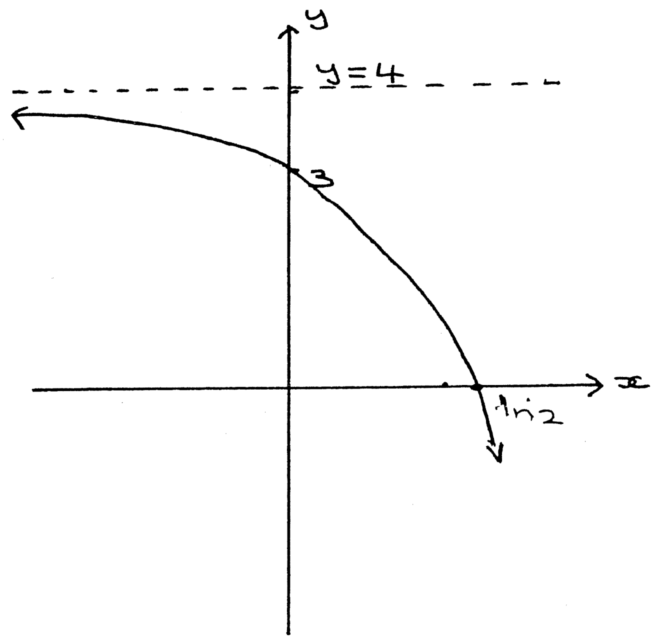
$x = \pm\sqrt{3}$



x) $y = f(e^x)$

$= 4 - (e^x)^2$

$= 4 - e^{2x}$



x-intercept $\Rightarrow y = 0$

$4 - e^{2x} = 0$

$e^{2x} = 4$

$2x = \ln 4$

$x = \ln 2$

b) $x^2 - xy + y^3 = 5$

$$2x - \left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3y^2 - x) = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{3y^2-x} \quad \text{at } x = 1,\ y = -2$$

$$= \frac{-2-2}{12-1}$$

$$= \frac{-4}{11}$$

c) i) $y = \sin^{-1}(e^x)$

domain $-1 \le e^x \le 1$

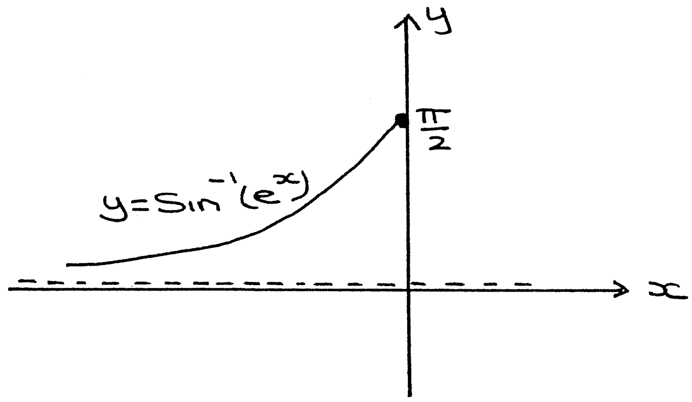
$0 \le e^x \le 1$

$x \le 0$

Range $0 < e^x \le 1$

$0 < \sin^{-1} e^x \le \frac{\pi}{2}$

$0 < y \le \frac{\pi}{2}$



Question 4

a) If $3-i$ is a root then $3+i$ is also a root by the conjugate root theorem

$$P(x) = x^3 + ax^2 + bx - 10$$

$$\alpha\beta\gamma = 10$$

$$(3-i)(3+i)\gamma = 10$$

$$10\gamma = 10$$

$$\gamma = 1$$

$\therefore$ The roots are $3-i,\ 3+i,\ 1$

$$\sum \alpha = -a$$

$$3-i+3+i+1 = -a$$

$$a = -7$$

$$\sum \alpha\beta = b$$

$$3-i+3+i+(3-i)(3+i) = b$$

$$b = 16$$

b) Let $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

$$P'(x) = 4x^3 + 3x^2 - 6x - 5$$

$$P''(x) = 12x^2 + 6x - 6$$

By the Multiple Root Theorem

$$P''(x) = P'(x) = P(x) = 0$$

$$12x^2 + 6x - 6 = 0$$

$$2x^2 + x - 1 = 0$$

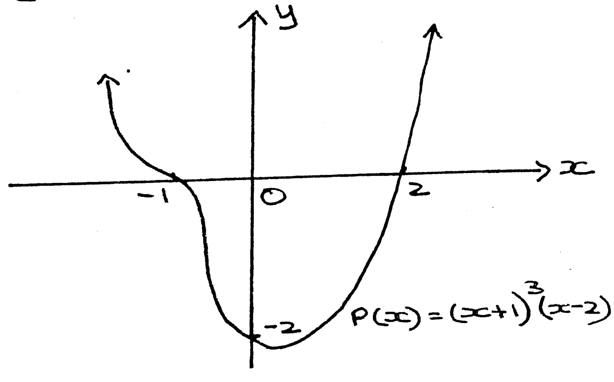
$$(2x-1)(x+1) = 0$$

$$x = \tfrac{1}{2}, -1$$

$\therefore P''(1) = P'(1) = P'(1) = 0$

$\therefore x = -1$ is a triple root

$\therefore P(x) = (x+1)^3(x-2)$



c) $\alpha+\beta+\gamma=-3$ —— ①

$\alpha^2+\beta^2+\gamma^2=29$ —— ②

$\alpha\beta\gamma \quad =-6$ —— ③

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$$

$$29=(-3)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$$

$$\alpha\beta+\alpha\gamma+\beta\gamma=-10 \quad \text{—— ④}$$

$\therefore$ $x^3+3x^2-10x+6=0$ is the required polynomial

d) i) $P(x)=x^4-5x^3+4x^2+2x-8$

$P(-1)=1+5+4-2-8$

$=0$

$\therefore$ $(x+1)$ is a factor

By synthetic division

$$\begin{array}{r|rrrrr} & 1 & -5 & 4 & 2 & -8 \\ -1 & 0 & -1 & 6 & -10 & 8 \\ \hline & 1 & -6 & 10 & -8 & 0 \end{array}$$

$\therefore$ $P(x)=x^4-5x^3+4x^2+2x-8$

$=(x+1)(x^3-6x^2+10x-8)$

$=(x+1)(x-4)(x^2-2x+2)$

$$\begin{array}{r|rrrr} & 1 & -6 & 10 & -8 \\ 4 & 0 & 4 & -8 & 8 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

(ii) $P(x)=(x+1)(x-4)(x^2-2x+1+1)$

$=(x+1)(x-4)[(x-1)^2-i^2]$

$=(x+1)(x-4)(x-1-i)(x-1+i)$

e) $x^3+2x^2+bx-16=0$ $\quad \alpha\beta=4$

$\alpha\beta\gamma=16$

$4\gamma=16$

$\gamma=4$

$\alpha\beta+\alpha\gamma+\beta\gamma=b$

$4+4(\alpha+\beta)=b$ —— ①

$\alpha+\beta+\gamma=-2$

$\alpha+\beta=-6$ —— ②

sub ② in ① $\quad 4+4(-6)=b$

$b=-20$

(ii) If α, β, γ satisfy $x^3+2x^2-20x-16=0$

Then $\alpha^2, \beta^2, \gamma^2$ satisfy

$$\left(x^{\frac{1}{2}}\right)^3+2\left(x^{\frac{1}{2}}\right)^2-20\left(x^{\frac{1}{2}}\right)-16=0$$

$$x^{\frac{3}{2}}+2x-20x^{\frac{1}{2}}-16=0$$

$$x^{\frac{3}{2}}-20x^{\frac{1}{2}}=16-2x$$

$$x^{\frac{1}{2}}(x-20)=16-2x$$

$$x(x-20)^2=(16-2x)^2$$

$$x(x^2-40x+400)=256-64x+4x^2$$

$$x^3-40x^2+400x-4x^2+64x-256=0$$

$$x^3-44x^2+464x-256=0$$

iii) $P(x) = x^3+2x^2-20x-16 = 0$

$$\alpha^3+2\alpha^2-20\alpha-16=0 \quad \text{——} \ 1$$

$$\beta^3+2\beta^2-20\beta-16=0 \quad \text{——} \ 2$$

$$\gamma^3+2\gamma^2-20\gamma-16=0 \quad \text{——} \ 3$$

$$\therefore \alpha^3+\beta^3+\gamma^3+2(\alpha^2+\beta^2+\gamma^2)-20(\alpha+\beta+\gamma)-48=0$$

$$\alpha^3+\beta^3+\gamma^3+2(44)-20(-2)-48=0$$

$$\alpha^3+\beta^3+\gamma^3=-80$$

NB $\alpha^2+\beta^2+\gamma^2 = \frac{-b}{a}$

$= 44$ from part (ii)

$\alpha+\beta+\gamma = \frac{-b}{a}$

$= 2$ from part (i)